The connection between distortion risk measures and ordered weighted averaging operators*

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Abstract

Distortion risk measures summarize the risk of a loss distribution by means of a single value. In fuzzy systems, the Ordered Weighted Averaging (OWA) and Weighted Ordered Weighted Averaging (WOWA) operators are used to aggregate a large number of fuzzy rules into a single value. We show that these concepts can be derived from the Choquet integral, and then the mathematical relationship between distortion risk measures and the OWA and WOWA operators for discrete and finite random variables is presented. This connection offers a new interpretation of distortion risk measures and, in particular, Value-at-Risk and Tail Value-at-Risk can be understood from a different perspective. The theoretical results are illustrated in an example.

1 Introduction

The relationship between two apparently unconnected worlds, risk theory and fuzzy systems, is investigated in this paper. Risk theory evaluates potential losses and is useful for decision making under probabilistic uncertainty. Broadly speaking, fuzzy logic is a form of reasoning based on the 'degree of truth' rather than on the binary true-false principle. But risk theory and fuzzy systems share a common core theoretical background. Both fields are related to the human psychological behavior under risk, ambiguity or uncertainty. The Expected Utility Theory by von Neumann and Morgenstern (1947) is one of the first attempts to provide a theoretical foundation to human behavior in decision-making, mainly based on setting up axiomatic preference relations of the decision maker. Similar theoretical approaches are, for instance, the Certainty-Equivalence Theory (Handa,

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1977), the Cumulative Prospect Theory (Kahneman and Tversky, 1979), the Theory of Anticipated Utility (Quiggin, 1982), the Dual Theory of Choice under Risk (Yaari, 1987), the Expected Utility without Subadditivity (Schmeidler, 1989) or the Cumulative Representation of Uncertainty (Tversky and Kahneman, 1992), where the respective axioms reflect different human behaviors or preference relations in decision-making under the unpredictable.

Previous attempts to link risk management and fuzzy logic approaches are mainly found in the literature on fuzzy systems. Most authors have focused on the application of fuzzy criteria to financial decision making (Engemann et al., 1996; Gil-Lafuente, 2005; Merigó and Casanovas, 2011), where some have smoothed financial series under fuzzy logic for prediction purposes (Yager and Filev, 1999; Yager, 2008). However, to our knowledge, the mathematical connections between these two worlds have not yet been provided for risk measurement applications.

In this paper we analyze the mathematical relationship between risk measurement and aggregation in fuzzy systems for discrete random variables. A risk measure quantifies the complexity of a random loss in one value that reflects the amount at risk. A key concept in fuzzy systems applications is the aggregation operator, which also allows to combine data into a single value. We show the relationship between the well-known distortion risk measures introduced by Wang (1996) and two specific aggregation operators, the Ordered Weighted Averaging (OWA) operator introduced by Yager (1988) and the Weighted Ordered Weighted Averaging (WOWA) operator introduced by Torra (1997).

Distortion risk measures, OWA and WOWA operators can be analyzed from the Theory of Measure. Classical measures are additive, and linked to the Lebesgue integral. When the additivity is relaxed, different measures and, hence, different integrals are derived. This is the case of non-additive measures (often called capacities as it was the name coined by Choquet, 1954), and we show that the link between distortion risk measures and OWA and WOWA operators is derived by means of the integral linked to capacities, i.e. the Choquet integral.

The paper is organized as follows. In section 2, we introduce risk theory concepts and fuzzy systems concepts. The relationship between distortion risk measures and aggregation operators is provided in section 3. An application with some classical risk measures is given in section 4. Finally, implications derived from these results are discussed in the conclusions.

2 Background and notation

In order to keep this article self-contained and to present the connection between two apparently distant theories, we need to introduce the notation and some basic definitions that may be well known for most readers.

2.1 Distortion risk measures

A recursive question in risk management is the suitability of risk measures, i.e. a good risk measure should properly reflect the borne risk. A lot of research aims to answer this question and most often it is addressed from an axiomatic point of view. In other words, risk measures are forced to satisfy suitable properties. Two well known groups of axiom-based risk measures are *coherent risk measures*, as stated by Artzner et al. (1999), and *distortion risk measures*, as introduced by Wang (1996); Wang et al. (1997). Concavity of the distortion function is the key element to define risk measures that belong to both families (Wang and Dhaene, 1998). Suggestions on new desirable properties for distortion risk measures are proposed in Balbas et al. (2009), while generalizations of this kind of risk measures can be found, among others, in Hürlimann (2006) and Wu and Zhou (2006).

The axiomatic research on risk measures has extensively been developed since seminal papers on coherent risk measures and distortion risk measures. Each set of axioms for risk measures corresponds to a particular behavior of decision makers under risk, ambiguity or uncertainty, as it has been shown, for instance, in Bleichrodt and Eeckhoudt (2006) and Denuit et al. (2006). Most often, those articles present the link to a theoretical foundation of human behavior explicitly. For example Wang (1996) show the connection between distortion risk measures and Yaari's Dual Theory of Choice Under Risk and Kaluszka and Krzeszowiec (2012) introduce the generalized Choquet integral premium principle and relate it to Kahneman and Tversky's Cumulative Prospect Theory.

Basic risk concepts are formally defined below. We need to start from that point to set up the notation used in section 3.

Definition 2.1 (Probability space). *A probability space is defined through three elements* $(\Omega, \mathcal{A}, \mathcal{P})$. *The sample space* Ω *is a set of the possible events of a random experiment,* $\mathcal{A}$ *is a family of the set of all subsets of* Ω *(denoted as* $\mathcal{A} \in \wp(\Omega)$*) with a* $\sigma-$*algebra structure, and the probability* $\mathcal{P}$ *is a mapping from* $\mathcal{A}$ *to* $[0,1]$ *such that* $\mathcal{P}(\Omega) = 1$, $\mathcal{P}(\emptyset) = 0$ *and* $\mathcal{P}$ *satisfies the* $\sigma - additivity$ *property.*

The probability space is finite if the sample space is finite, $\Omega = \{\varpi_1, \varpi_2, ..., \varpi_n\}$. Then $\wp(\Omega)$ is the $\sigma - algebra$, which is denoted as 2^{Ω}. In the rest of the article, N instead of Ω will be used when referring to finite probability spaces. Hence, the notation will be $\left(N, 2^N, \mathcal{P}\right)$.

Definition 2.2 (Random variable). *Let* $(\Omega, \mathcal{A}, \mathcal{P})$ *be a probability space. A random variable* X *is a mapping from* Ω *to* $\mathbb{R}$ *such that* $X^{-1}((-\infty, x]) := \{\varpi \in \Omega : X(\varpi) \leq x\} \in \mathcal{A}$, $\forall x \in \mathbb{R}$.

A random variable X is discrete if $X(\Omega)$ is a finite set or a numerable set without cumulative points, i.e. $X(\Omega)$ is $\{x_1, x_2, ..., x_n, ...\}$.

Definition 2.3 (Distribution function of a random variable). *Let* X *be a random variable. The distribution function of* X*, denoted by* F_X*, is defined by*

$$F_X(x) = \mathcal{P}\left(X^{-1}((-\infty, x])\right) = \mathcal{P}(\{\varpi \in \Omega : X(\varpi) \leq x\}) \equiv \mathcal{P}(X \leq x).$$

F_X is non-decreasing, right-continuous and $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$. The survival function of X, denoted by S_X, is defined by $S_X(x) = 1 - F_X(x)$, , $x \in \mathbb{R}$. Note that the domain of the distribution function and the survival function is $\mathbb{R}$ even if X is a discrete random variable. In other words, F_X and S_X are defined for $X(\Omega) = \{x_1, x_2, ..., x_n, ...\}$ but also for any $x \in \mathbb{R} \backslash \{x_1, x_2, ..., x_n, ...\}$.

Definition 2.4 (Risk measure)**.** *Let Γ be the set of all random variables defined for a given probability space $(\Omega, \mathcal{A}, \mathcal{P})$. A risk measure is a mapping ρ from Γ to $\mathbb{R}$, so that $\rho(X)$ is a real value for each $X \in \Gamma$.*

Definition 2.5 (Distortion risk measure)**.** *Let $g : [0,1] \to [0,1]$ be a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ (we will call g a distortion function). A distortion risk measure associated to distortion function g is defined by*

$$\rho_g(X) := -\int_{-\infty}^{0} [1 - g(S_X(x))]\, dx + \int_{0}^{+\infty} g(S_X(x))\, dx$$

The simplest distortion risk measure is the mathematical expectation, which is obtained when the distortion function is the identity (see Denuit et al., 2005). The two most widely used distortion risk measures are the Value-at-Risk (VaR_α) and the Tail Value-at-Risk ($TVaR_\alpha$). Broadly speaking, the VaR_α corresponds to a high percentile of the distribution function. The $TVaR_\alpha$ is the expected value over this percentile [1] if the random variable is continuous. The former pursues to answer what is the maximum loss that can be suffered with a certain confidence level α, where $\alpha \in (0,1)$. The latter evaluates what is the expected loss if the loss is larger than the VaR_α for a given confidence level. Both risk measures are distortion risk measures with associated distortion functions shown in Table 2.1. Unlike the VaR_α, the distortion function associated to the $TVaR_\alpha$ is concave and, then, the $TVaR_\alpha$ is a *coherent* risk measure in the sense of Artzner et al. (1999). Basically, it means that $TVaR_\alpha$ is sub-additive (see Acerbi and Tasche, 2002) while the VaR_α is not.

Table 2.1: Correspondence between risk measures and distortion functions

Risk measure	Distortion function $g(x)$
VaR_α	$\psi_\alpha(x) = \left\{ \begin{array}{ll} 0 & \text{if} \quad x \leq 1-\alpha \\ 1 & \text{if} \quad x > 1-\alpha \end{array} \right\} = \mathbb{1}_{(1-\alpha,1]}(x)$
$TVaR_\alpha$	$\gamma_\alpha(x) = \left\{ \begin{array}{ll} \dfrac{x}{1-\alpha} & \text{if} \quad x \leq 1-\alpha \\ 1 & \text{if} \quad x > 1-\alpha \end{array} \right\} = \min\left\{ \dfrac{x}{1-\alpha}, 1 \right\}$

[1] We consider $TVaR_\alpha$ as defined in Denuit et al. (2005). That is, $TVaR_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_\delta(X)\, d\delta$.

2.2 The OWA and WOWA operators and the Choquet integral

Aggregation operators (or aggregation functions) have extensively been used as a natural form to combine inputs into a single value. These inputs are typically interpreted as degrees of membership in fuzzy theory, degrees of preference, strength or evidence, or support of a hypothesis. Aggregation operators are considered when a multi-criteria decision-making problem, connectives in fuzzy logic or group decision-making problems are faced up, because this functions aim at summarizing data into a single value according to specific aggregation criteria. Let us denote by $\overline{\mathbb{R}} = [-\infty, +\infty]$ the extended real line, and by $\mathbb{I}$ any type of interval in $\overline{\mathbb{R}}$ (open, closed, with extremes being $\mp\infty$,...). Following Grabisch et al. (2011), an aggregation operator is defined as follows.

Definition 2.6 (Aggregation operator)**.** *An aggregation operator in* $\mathbb{I}^n$ *is a function* $F^{(n)} : \mathbb{I}^n \longrightarrow \mathbb{I}$*, that is non-decreasing in each variable; fulfills the following boundary conditions:* $\inf_{\vec{x}\in\mathbb{I}^n} F^{(n)}(\vec{x}) = \inf \mathbb{I}$*,* $\sup_{\vec{x}\in\mathbb{I}^n} F^{(n)}(\vec{x}) = \sup \mathbb{I}$ *and* $F^{(1)}(x) = x$ *for all* $x \in \mathbb{I}$*.*

Some basic aggregation operators are displayed in Table 2.2.

There is a huge amount of literature on aggregation operators and its applications (see, for example, Beliakov et al., 2007; Torra and Narukawa, 2007; Grabisch et al., 2009, 2011). Despite the large number of aggregation operators, we focus on the OWA operator and on the WOWA operator. Several reasons lead us to this selection. The OWA operator has been extensively applied in the context of decision making under uncertainty because it provides a unified formulation for the optimistic, the pessimistic, the Laplace and the Hurwicz criteria (Yager, 1993), and there are also some interesting generalizations (Yager et al., 2011). The WOWA operator combines the OWA operator with the concept of weighted average, where weights are a mechanism to include expert opinion on the accuracy of information. This operator is closely linked to distorted probabilities.

2.2.1 Ordered Weighted Averaging operator

The OWA operator is an aggregation operator that provides a parameterized family of aggregation operators offering a compromise between the minimum and the maximum aggregation functions (Yager, 1988). It can be defined as follows [2]

Definition 2.7 (OWA operator)**.** *Let* $\vec{w} = (w_1, w_2, ..., w_n) \in [0,1]^n$ *such that* $\sum_{i=1}^{n} w_i = 1$*. The Ordered Weighted Average (OWA) operator with respect to* $\vec{w}$ *is a mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *defined by* $OWA_{\vec{w}}(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_{\sigma(i)} \cdot w_i$*, where* σ *is a permutation of* $(1, 2, ..., n)$ *such that* $x_{\sigma(1)} \leq x_{\sigma(2)} \leq ... \leq x_{\sigma(n)}$*, i.e.* $x_{\sigma(i)}$ *is the i-th smallest value of* $x_1, x_2, ..., x_n$*.*

[2]Unlike the original definition provided by Yager (1988), we consider an ascending order in $\vec{x}$ instead of a decreasing one. This definition is convenient from the risk management perspective since $\vec{x}$ may be a set of losses and losses are usually ordered in ascending order. This approach is not new. The relationship between the ascending OWA and the descending OWA operators is provided by Yager (1993).

Table 2.2: Basic $F^{(n)}$ aggregation operators

Name	Mathematical expression	Type of interval $\mathbb{I}$
Arithmetic mean	$AM(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$	Arbitrary $\mathbb{I}$. If $\mathbb{I} = \overline{\mathbb{R}}$, the convention $+\infty+(-\infty) = -\infty$ is often considered.
Product	$\Pi(\vec{x}) = \prod_{i=1}^{n}(x_i)$	$\mathbb{I} \in \{\|0,1\|, \|0,+\infty\|, \|1,+\infty\|\}$, where $\|a,b\|$ means any kind of interval, with boundary points a and b, and with the convention $0 \cdot (+\infty) = 0$.
Geometric mean	$GM(\vec{x}) = \left(\prod_{i=1}^{n}(x_i)\right)^{1/n}$	$\mathbb{I} \subseteq [0,+\infty]$, with the convention $0 \cdot (+\infty) = 0$.
Minimum function	$Min(\vec{x}) = \min\{x_1, x_2, ..., x_n\}$	Arbitrary $\mathbb{I}$.
Maximum function	$Max(\vec{x}) = \max\{x_1, x_2, ..., x_n\}$	Arbitrary $\mathbb{I}$.
Sum function	$\sum(\vec{x}) = \sum_{i=1}^{n} x_i$	$\mathbb{I} \in \{\|0,+\infty\|, \|-\infty,0\|, \|-\infty,+\infty\|\}$, with the convention $+\infty + (-\infty) = -\infty$.
k-order statistics	$OS_k(\vec{x}) = x_j$, $k \in \{1,...,n\}$ where x_j is such that $\#\{i \mid x_i \leq x_j\} \geq k$ and $\#\{i \mid x_i > x_j\} < n-k$	Arbitrary $\mathbb{I}$.
k-th projection	$P_k(\vec{x}) = x_k$, $k \in \{1,...,n\}$	Arbitrary $\mathbb{I}$.

Source: Grabisch et al. (2011), $\vec{x}$ denotes $(x_1, x_2, ..., x_n)$

The OWA operator is commutative, monotonic and idempotent, and it is lower-bounded by the minimum and upper-bounded by the maximum operators. Commutativity is referred to any permutation of the components of $\vec{x}$. That is, if the $OWA_{\vec{w}}$ operator is applied to any $\vec{y}$ such that $y_i = x_{r(i)}$ for all i, and r is any permutation of $(1, ..., n)$, then $OWA_{\vec{w}}(\vec{y}) = OWA_{\vec{w}}(\vec{x})$. Monotonicity means that if $x_i \geq y_i$ for all i, then $OWA_{\vec{w}}(\vec{x}) \geq OWA_{\vec{w}}(\vec{y})$. Idempotency assures that if $x_i = a$ for all i, then $OWA_{\vec{w}}(\vec{x}) = a$. The OWA operator accomplishes the boundary conditions because it is delimited by the minimum and the maximum functions, i.e. $\min_{i=1,...,n}\{x_i\} \leq OWA_{\vec{w}}(\vec{x}) \leq \max_{i=1,...,n}\{x_i\}$.

The $OWA_{\vec{w}}$ is unique with respect to the vector $\vec{w}$ (the proof is provided in the Appendix). The characterization of the weighting vector $\vec{w}$ is often made by means of the *degree of orness* measure (Yager, 1988).

Definition 2.8 (Degree of orness of an OWA operator)**.** *Let $\vec{w} \in [0,1]^n$ such that $\sum_{i=1}^{n} w_i = 1$, the degree of orness of $OWA_{\vec{w}}$ is defined by*

$$orness\left(OWA_{\vec{w}}\right) = \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot w_i$$

Note that the degree of orness represents the level of aggregation preference between the minimum and the maximum that is considered when $\vec{w}$ is fixed. The degree of orness can be understood as the value that the OWA operator returns when it is applied to $\vec{x^*} = \left(\frac{0}{n-1}, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, \frac{n-1}{n-1}\right)$. In other words, $orness\left(OWA_{\vec{w}}\right) = OWA_{\vec{w}}\left(\vec{x^*}\right)$. The fact that $orness\left(OWA_{\vec{w}}\right) \in [0,1]$ follows from $\vec{x^*}, \vec{w} \in [0,1]^n$. If $\vec{w} = (1, 0, ..., 0)$, then $OWA_{\vec{w}} \equiv Min$ and $orness\left(Min\right) = 0$. Conversely, if $\vec{w} = (0, 0, ..., 1)$, then $OWA_{\vec{w}} \equiv Max$ and $orness\left(Max\right) = 1$. And when $\vec{w}$ is such that $w_i = \frac{1}{n}$ for all i, then $OWA_{\vec{w}}$ is the arithmetic mean and its degree of orness is 0.5. As we will see later orness is closely related the α level chosen in risk measures.

Alternatively to the degree of orness, other measures can be used to characterize the weighting vector, such as the *entropy of dispersion* (Yager, 1988) based on the Shannon entropy (Shannon, 1948) and the *divergence of the weighting vector* (Yager, 2002). A summary of these and additional measures is found in Torra and Narukawa (2007, Ch.7).

The OWA operator has been extended and generalized in different ways. For example, Xu and Da (2002) introduced the uncertain OWA (UOWA) operator in order to deal with imprecise information, Merigó and Gil-Lafuente (2009) developed a generalization by using induced aggregation operators and quasi-arithmetic means called the induced quasi-OWA (Quasi-IOWA) operator and Yager (2010) introduced a new approach for using norms in the OWA operator. Although it is out of the scope of this paper, the OWA operator is also related to the linguistic quantifiers introduced by Zadeh (1985), where a class of them may be interpreted as distortion functions.

2.2.2 Weighted Ordered Weighted Averaging operator

The WOWA operator is the aggregation function introduced by Torra (1997). This operator unifies in the same formulation the weighted mean function and the OWA operator in the following way [3].

Definition 2.9 (WOWA operator)**.** *Let $\vec{v} = (v_1, v_2, ..., v_n) \in [0,1]^n$ and $\vec{q} = (q_1, q_2, ..., q_n) \in [0,1]^n$ such that $\sum_{i=1}^n v_i = 1$ and $\sum_{i=1}^n q_i = 1$. The Weighted Ordered Weighted Average (WOWA) operator with respect to $\vec{v}$ and $\vec{q}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by*

$$WOWA_{h,\vec{v},\vec{q}}(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_{\sigma(i)} \cdot \left[h\left(\sum_{j \in A_{\sigma,i}} q_j \right) - h\left(\sum_{j \in A_{\sigma,i+1}} q_j \right) \right]$$

where σ is a permutation of $(1, 2, ..., n)$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq ... \leq x_{\sigma(n)}$, $A_{\sigma,i} = \{\sigma(i), ..., \sigma(n)\}$ and $h : [0,1] \to [0,1]$ is a non-decreasing function such that $h(0) := 0$ and $h\left(\frac{i}{n}\right) := \sum_{j=n-i+1}^{n} v_j$; and h is linear if the points $\left(\frac{i}{n}, \sum_{j=n-i+1}^{n} v_j\right)$ lie on a straight line.

Note that this definition implies that weights v_i can be expressed as $v_i = h\left(\frac{n-i+1}{n}\right) - h\left(\frac{n-i}{n}\right)$ and that $h(1) = 1$.

The WOWA operator generalizes the OWA operator. Given a $WOWA_{\vec{v},\vec{q}}$ operator on $\mathbb{R}^n$ with associated h function, if we consider

$$w_i := h\left(\sum_{j \in A_{\sigma,i}} q_j \right) - h\left(\sum_{j \in A_{\sigma,i+1}} q_j \right)$$

and $OWA_{\vec{w}}$ where $\vec{w} = (w_1, ..., w_n)$, then the following equality holds $WOWA_{\vec{v},\vec{q}} = OWA_{\vec{w}}$. As it can easily be shown, vector $\vec{w}$ satisfies the following conditions:

(i) $\vec{w} \in [0,1]^n$;

(ii) $\sum_{i=1}^{n} w_i = 1$;

Condition *(i)* is easily shown. Let us denote $s_i = \sum_{j \in A_{\sigma,i}} q_j$ and $s_{n+1} := 0$. Hence, $s_i \geq s_{i+1}$ for all i due to the fact that $A_{\sigma,i} \supseteq A_{\sigma,i+1}$ and that $q_j \geq 0$. Then $h(s_i) \geq h(s_{i+1})$ since h is a non-decreasing function. Finally, as $s_i \in [0,1]$ and $h(s) \in [0,1]$ for all $s \in [0,1]$, then it follows that $w_i = h(s_i) - h(s_{i+1}) \in [0,1]$ for all i.

To prove condition *(ii)*, note that $A_{\sigma,1} = N$, $\sum_{j \in N} q_j = 1$ and that $h(1) = 1$ and $h(0) = 0$, then $\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} (h(s_i) - h(s_{i+1})) = h(s_1) - h(s_{n+1}) = 1 - 0 = 1$.

[3] In the original definition provided by Torra (1997) $\vec{x}$ components are in descending order, while we use ascending order. An additional subindex regarding the h function is also introduced.

Remark 1

Let us analyze the particular case when OWA and WOWA operators provide the expectation of random variables. Suppose that X is a discrete random variable that takes n different values and $\vec{x} \in \mathbb{R}^n$ is the vector of values where the components are in ascending order. Let us consider vector $\vec{w} \in [0,1]^n$ such that $\sum_{i=1}^{n} w_i = 1$ and vector $\vec{p} \in [0,1]^n$ refers to the probabilities of the components of $\vec{x}$. Obviously, it holds that $OWA_{\vec{p}}(\vec{x}) = \mathbb{E}(X)$. Then, $OWA_{\vec{w}}(\vec{x})$ is equal to $\mathbb{E}(X)$ if and only if $\vec{w} = \vec{p}$, due to the uniqueness of the OWA operator. Given a h function and $\vec{q} = \vec{p}$, the $WOWA_{h,\vec{v},\vec{q}}(\vec{x})$ may be expressed as

$$\begin{aligned} WOWA_{h,\vec{v},\vec{p}}(\vec{x}) = & \sum_{i=1}^{n} x_i \cdot \left[h\left(\sum_{j=i}^{n} p_j\right) - h\left(\sum_{j=i+1}^{n} p_j\right)\right] = \\ = & \sum_{i=1}^{n} x_i \cdot \left[h\left(S_X\left(x_{i-1}\right)\right) - h\left(S_X\left(x_i\right)\right)\right] \end{aligned}$$

If h is the identity function then $WOWA_{h,\vec{v},\vec{p}}(\vec{x}) = \mathbb{E}(X)$ because of $S_X(x_{i-1}) - S_X(x_i) = p_i$ for all i (with the convention $x_0 := -\infty$).

Remark 2

Note that if X is discrete and uniformly distributed then $S_X(x_{i-1}) = \dfrac{n-i+1}{n}$, and hence $h(S_X(x_{i-1})) = h\left(\dfrac{n-i+1}{n}\right) = \sum_{j=i}^{n} v_j$. This remark is helpful to interpret the WOWA operator from the perspective of risk measurement. In the WOWA operator the subjective opinion of experts may be represented by vector $\vec{v}$. Let us suppose that no information regarding the distribution function of a discrete and finite random variable X is available. If we assume that X is discrete and uniformly distributed, then vector $\vec{v}$ directly consists of the subjective probabilities of occurrence of the components x_i according to the expert opinion. Another possible point of view in this case is that $\vec{v}$ represents the subjective importance that the expert give to each x_i.
Nonetheless, note that the fact that the domain of the survival function is $\mathbb{R}$ implies that the selected h function is important from risk measurement point of view. The relevance of the h function is smaller in fuzzy systems.

2.2.3 The Choquet integral

The Choquet integral has become a familiar concept to risk management experts since it was introduced by Wang (1996) in the definition of *distortion risk measures*. OWA and WOWA operators can also be defined based on the concept of Choquet integral. In this subsection we follow Grabisch et al. (2011) to provide several definitions which are needed the section 3.

Definition 2.10 (Capacity). *Let* $N := \{m_1, ..., m_n\}$ *be a finite set and* $2^N := \wp(N)$ *be the set of all subsets of* N*. A capacity or a fuzzy measure on* N *is a mapping from* 2^N *to* $[0,1]$ *which satisfies*

(i) $\mu(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$*, for any* $A, B \in 2^N$ *(monotonicity).*

If $\mu(N) = 1$*, then we say that* μ *satisfies normalization, which is a frequently required property.*

Definition 2.11 (Dual capacity). *Let* μ *be a capacity on* N*. Its dual or conjugate capacity* $\bar{\mu}$ *is a capacity on* N *defined by*

$$\bar{\mu}(A) = \mu(N) - \mu(\bar{A}),$$

where $\bar{A} = N \backslash A$ *(i.e.,* $\bar{A}$ *is the set of all the elements in* N *that do not belong to* A*).*

If we consider a finite probability space $(N, 2^N, \mathcal{P})$, note that the probability $\mathcal{P}$ is a capacity (or a fuzzy measure) on N that satisfies normalization. In addition, $\mathcal{P}$ is its own dual capacity.

Definition 2.12 (Choquet integral for discrete positive functions). *Let* μ *be a capacity on* N*, and* $f : N \to [0, +\infty)$ *be a function. Let* σ *be a permutation of* $(1, ..., n)$*, such that* $f(m_{\sigma(1)}) \leq f(m_{\sigma(2)}) \leq ... \leq f(m_{\sigma(n)})$*, and* $A_{\sigma,i} = \{m_{\sigma(i)}, ..., m_{\sigma(n)}\}$*, with* $A_{\sigma,n+1} = \emptyset$*. The Choquet integral of* f *with respect to* μ *is defined by*

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} f(m_{\sigma(i)}) \left(\mu(A_{\sigma,i}) - \mu(A_{\sigma,i+1})\right).$$

If we let $f(m_{\sigma(0)}) := 0$, then an equivalent expression for the definition of the Choquet integral is $\mathcal{C}_\mu(f) = \sum_{i=1}^{n} \left[f(m_{\sigma(i)}) - f(m_{\sigma(i-1)})\right] \mu(A_{\sigma,i})$.

The concept of degree of orness introduced for the OWA operator may be extended to the case of the Choquet integral for positive functions as:

$$orness(\mathcal{C}_\mu) := \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot \left(\mu(A_{id,i}) - \mu(A_{id,i+1})\right). \tag{2.1}$$

We will concentrate on three particular capacities. The first one, denoted as μ_*, is such that $\mu_*(A) = 0$ if $A \neq N$ and $\mu_*(N) = 1$. In this case, $\mathcal{C}_{\mu_*} \equiv Min$ and we find through expression (2.1) that $orness(Min) = 0$. The second case, denoted as μ^*, is such that $\mu^*(A) = 0$ if $A \neq \{n\}$ and $\mu^*(\{n\}) = 1$. In this situation, $\mathcal{C}_{\mu^*} \equiv Max$ and, as expected, we get that $orness(Max) = 1$. Finally, we consider capacities $\mu^{\#}$ such that $\mu^{\#}(A)$ solely depends on the cardinality of A for all $A \subseteq N$. Then $\mu^{\#}(A_{\sigma,i}) - \mu^{\#}(A_{\sigma,i+1})$

is defined by i. If we denote by $w_i := \mu^{\#}(A_{\sigma,i}) - \mu^{\#}(A_{\sigma,i+1})$ for all i, it follows that $\mathcal{C}_{\mu^{\#}}$ is equal to $OWA_{\vec{w}}$. In the particular case where $\mu^{\#}(A) = \frac{\#A}{n}$ for any $A \subseteq N$, then $w_i = \frac{n-(i-1)}{n} - \frac{n-i}{n} = \frac{1}{n}$. So, in this situation $\mathcal{C}_{\mu^{\#}}$ is the arithmetic mean, and we can verify that $orness\left(\mathcal{C}_{\mu^{\#}}\right) = 0.5$ in the following way:

$$\begin{aligned} orness\left(\mathcal{C}_{\mu^{\#}}\right) &= \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)\cdot\left(\mu^{\#}(A_{id,i}) - \mu^{\#}(A_{id,i+1})\right) = \\ &= \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n-1}\right)\cdot\frac{1}{n} - \frac{1}{n-1} = \\ &= \frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{n\cdot(n+1)}{2} - \frac{1}{n-1} = \frac{n-1}{2}\cdot\frac{1}{n-1} = \frac{1}{2}. \end{aligned} \tag{2.2}$$

In order to be able to work with negative functions, the Choquet integral of such functions needs to be defined for that case. Below we define the asymmetric Choquet integral, which is the classical extension from real-valued positive functions to negative functions. Nonetheless, symmetric extensions have gained an increasing interest (Kojadinovic et al., 2005; **?**, see).

Definition 2.13 (Asymmetric Choquet integral for discrete negative functions)**.** *Let $f : N \to (-\infty, 0]$ be a function, μ a capacity on N and $\bar{\mu}$ its dual capacity. The asymmetric Choquet integral of f with respect to μ is defined by $\mathcal{C}_{\mu}(f) := -\mathcal{C}_{\bar{\mu}}(-f)$.*

If f is a function from N to $\mathbb{R}$ and μ is a capacity on N, and we denote by $f^{+}(m_i) = max\{f(m_i), 0\}$ and $f^{-}(m_i) = min\{f(m_i), 0\}$, then the Choquet integral of f with respect to μ is defined by

$$\mathcal{C}_{\mu}(f) = \mathcal{C}_{\mu}\left(f^{+}\right) + \mathcal{C}_{\mu}\left(f^{-}\right) = \mathcal{C}_{\mu}\left(f^{+}\right) - \mathcal{C}_{\bar{\mu}}\left(-f^{-}\right). \tag{2.3}$$

3 The relationship between distortion risk measures, OWA and WOWA operators

Three main results for discrete random variables are provided in this section. First, the equivalence between the Choquet integral and a distortion risk measure is shown, when the distortion risk measure is fixed on a finite probability space. Second, the link between this distortion risk measure and a set of OWA operators is provided. And, third, the relationship between the fixed distortion risk measure and a set of WOWA operators is given. Finally, we show that the degree of orness of the VaR_{α} and $TVaR_{\alpha}$ risk measures may be defined as a function of the confidence level when the random variable is given. To our knowledge, some of these results provide a new insight into the way classical risk quantification is understood, as it can now be viewed as a weighted aggregation.

The first result is known by the risk community for arbitrary random variables almost since the inception of distortion risk measures (see Wang, 1996), and has lead to many interesting results. For example, the concept of Choquet pricing and its associated equilibrium conditions (De Waegenaere et al., 2003); the study of stochastic comparison of distorted variability measures (Sordo and Suarez-Llorens, 2011); or the conditions for optimal behavioral insurance (Sung et al., 2011) and the analysis of competitive insurance markets in the presence of ambiguity (Anwar and Zheng, 2012). On the other side, the relationship between the WOWA operator and the Choquet integral is also known by the fuzzy systems community (Torra, 1998), as well as the relationship between distorted probabilities and aggregation operators (see, for example, Honda and Okazaki, 2005). Through the results shown in this section we provide a presentation that allows both fields to share their knowledge and we can benefit from that connection.

Proposition 3.1. *Let* $\left(N, 2^N, \mathcal{P}\right)$ *be a finite probability space, and let* X *be a discrete finite random variable defined on this space. Let* $g:[0,1]\to[0,1]$ *be a distortion function, and let* ρ_g *be the associated distortion risk measure. Then, it follows that*

$$\mathcal{C}_{g\circ\mathcal{P}}\left(X\right)=\rho_g\left(X\right).$$

Proof. Let $N=\{\varpi_1,...,\varpi_n\}$ for some $n\geq 1$ and let us suppose that we can write $X\left(N\right)=\{x_1,...,x_n\}$, with $X\left(\{\varpi_i\}\right)=x_i$, and such that $x_i<x_j$ if $i<j$; additionally, let $k\in\{1,...,n\}$ be such that $x_i<0$ if $i=\{1,...,k-1\}$ and $x_i\geq 0$ if $i=\{k,,...,n\}$. In order to obtain the Choquet integral of X, a capacity μ defined on N is needed. As previously indicated, $\mathcal{P}$ is a capacity on N that satisfies normalization, although it is not the one that we need.

Since g is a distortion function, $\mu:=g\circ\mathcal{P}$ is another capacity on N that satisfies normalization: $\mu\left(\emptyset\right)=g\left(\mathcal{P}\left(\emptyset\right)\right)=g(0)=0$, $\mu\left(N\right)=g\left(\mathcal{P}\left(N\right)\right)=g(1)=1$, and if $A\subseteq B$, the fact that $\mathcal{P}\left(A\right)\leq\mathcal{P}\left(B\right)$ and the fact that g is non-decreasing imply that $\mu\left(A\right)\leq\mu\left(B\right)$.

Regarding X^+, the permutation $\sigma=id$ on $(1,...,k-1,k,...,n)$ is such that $x^+_{\sigma(i)}\leq x^+_{\sigma(i+1)}$ for all i or, in other words, $x^+_1\leq x^+_2\leq...\leq x^+_{k-1}\leq x^+_k\leq x^+_{k+1}\leq...\leq x^+_n$. Then, $A_{\sigma,i}=\{\varpi_i,...,\varpi_n\}$ and taking into account $x^+_i=0\ \forall i<k$, we can write $\mathcal{C}_{g\circ\mathcal{P}}\left(X^+\right)$ as

$$\mathcal{C}_{g\circ\mathcal{P}}\left(X^+\right)=\sum_{i=1}^{n}\left(x^+_i-x^+_{i-1}\right)\left(g\circ\mathcal{P}\right)\left(A_{\sigma,i}\right)=\sum_{i=k}^{n}\left(x^+_i-x^+_{i-1}\right)g\left(\sum_{j=i}^{n}p_j\right). \tag{3.1}$$

Additionally, permutation s on $(1,...,k-1,k,...,n)$ such that $s\left(i\right)=n+1-i$, satisfies $-x^-_{s(i)}\leq -x^-_{s(i+1)}$ for all i, so $-x^-_n\leq -x^-_{n-1}\leq...\leq -x^-_k\leq -x^-_{k-1}\leq -x^-_{k-2}\leq ...\leq -x^-_1$. We have $A_{s,i}=\left\{\varpi_{s(i)},...,\varpi_{s(n)}\right\}=\{\varpi_{n+1-i},...,\varpi_1\}$ and, therefore, $\bar{A}_{s,i}=$

$\{\varpi_{n+2-i}, ..., \varpi_n\}$. Taking into account that $x_i^- = 0 \; \forall i \geq k$, we can write $\mathcal{C}_{\overline{g \circ \mathcal{P}}}(-X^-)$ as

$$\begin{aligned} \mathcal{C}_{\overline{g \circ \mathcal{P}}}(-X^-) = & \sum_{i=1}^{n} \left(-x_{s(i)}^- + x_{s(i-1)}^-\right) \left(\overline{g \circ \mathcal{P}}\right)(A_{s,i}) = \\ = & \sum_{i=1}^{n} \left(-x_{n+1-i}^- + x_{n+2-i}^-\right) \left(\overline{g \circ \mathcal{P}}\right)(A_{s,i}) = \\ = & \sum_{i=n}^{1} \left(-x_i^- + x_{i+1}^-\right) \left(\overline{g \circ \mathcal{P}}\right)(A_{s,n+1-i}) = \\ = & \sum_{i=n}^{1} \left(-x_i^- + x_{i+1}^-\right) \left[1 - (g \circ \mathcal{P})\left(\bar{A}_{s,n+1-i}\right)\right] = \\ = & \sum_{i=n}^{1} \left(-x_i^- + x_{i+1}^-\right) \left[1 - (g \circ \mathcal{P})\left(\{\varpi_{i+1}, ..., \varpi_n\}\right)\right] = \\ = & \sum_{i=k-1}^{1} \left(x_{i+1}^- - x_i^-\right) \left[1 - g\left(\sum_{j=i+1}^{n} p_j\right)\right]. \end{aligned} \tag{3.2}$$

Expressions (3.1) and (3.2) lead to

$$\begin{aligned} \mathcal{C}_{g \circ \mathcal{P}}(X) = & \; \mathcal{C}_{g \circ \mathcal{P}}(X^+) - \mathcal{C}_{\overline{g \circ \mathcal{P}}}(-X^-) = \\ = & -\sum_{i=1}^{k-1} \left(x_{i+1}^- - x_i^-\right) \left[1 - g\left(\sum_{j=i+1}^{n} p_j\right)\right] + \sum_{i=k}^{n} \left(x_i^+ - x_{i-1}^+\right) g\left(\sum_{j=i}^{n} p_j\right) = \\ = & -\sum_{i=2}^{k} (x_i - x_{i-1}) \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] + x_k \left[1 - g\left(\sum_{j=k}^{n} p_j\right)\right] + \\ & + \sum_{i=k+1}^{n} (x_i - x_{i-1})\, g\left(\sum_{j=i}^{n} p_j\right) + x_k g\left(\sum_{j=k}^{n} p_j\right) = \\ = & -\sum_{i=2}^{k} (x_i - x_{i-1}) \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] + x_k + \sum_{i=k+1}^{n} (x_i - x_{i-1})\, g\left(\sum_{j=i}^{n} p_j\right). \end{aligned} \tag{3.3}$$

Now consider $\rho_g(X)$ as in definition 2.5, and note that random variable X is defined on the probability space $(N, 2^N, \mathcal{P})$. Given the properties of Riemann's integral, if we define $x_0 := -\infty$ and $x_{n+1} := +\infty$, then the distortion risk measure can be written as

$$\begin{aligned} \rho_g(X) = & -\left[\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} [1 - g(S_X(x))]dx - \int_0^{x_k} [1 - g(S_X(x))]dx\right] + \\ & + \int_0^{x_k} g(S_X(x))dx + \sum_{i=k+1}^{n+1} \int_{x_{i-1}}^{x_i} g(S_X(x))dx. \end{aligned} \tag{3.4}$$

If we consider $x \in [x_{i-1}, x_i)$, then $F_X(x) = \sum_{j=1}^{i-1} p_j$, since $F_X(x) = \mathcal{P}(X \leq x)$ and

$S_X(x) = 1 - \sum_{j=1}^{i-1} p_j = \sum_{j=i}^{n} p_j$. Given that the distortion function g is such that $g(0) = 0$ and $g(1) = 1$, expression (3.4) can be rewritten as

$$\begin{aligned}
\rho_g(X) = & -\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] dx + \int_{0}^{x_k} \left[1 - g\left(\sum_{j=k}^{n} p_j\right)\right] dx \\
& + \int_{0}^{x_0} g\left(\sum_{j=k}^{n} p_j\right) dx + \sum_{i=k+1}^{n+1} \int_{x_{i-1}}^{x_i} g\left(\sum_{j=i}^{n} p_j\right) dx = \\
= & -\int_{-\infty}^{x_1} [1 - g(1)]\, dx - \sum_{i=2}^{k} \int_{x_{i-1}}^{x_i} \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] dx + \\
& + \int_{0}^{x_k} \left[1 - g\left(\sum_{j=k}^{n} p_j\right)\right] dx + \int_{0}^{x_k} g\left(\sum_{j=k}^{n} p_j\right) dx + \\
& + \sum_{i=k+1}^{n} \int_{x_{i-1}}^{x_i} g\left(\sum_{j=i}^{n} p_j\right) dx + \int_{x_n}^{+\infty} g(0)\, dx = \\
= & -\sum_{i=2}^{k} (x_i - x_{i-1}) \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] + x_k \left[1 - g\left(\sum_{j=k}^{n} p_j\right) + g\left(\sum_{j=k}^{n} p_j\right)\right] + \\
& + \sum_{i=k+1}^{n} (x_i - x_{i-1})\, g\left(\sum_{j=i}^{n} p_j\right) = \\
= & -\sum_{i=2}^{k} (x_i - x_{i-1}) \left[1 - g\left(\sum_{j=i}^{n} p_j\right)\right] + x_k + \sum_{i=k+1}^{n} (x_i - x_{i-1})\, g\left(\sum_{j=i}^{n} p_j\right).
\end{aligned} \tag{3.5}$$

And then the proof is finished because $\rho_g(X) = \mathcal{C}_{g \circ \mathcal{P}}(X)$ using (3.5) and (3.3). □

Let us present $\mathcal{C}_{g \circ \mathcal{P}}(X)$ in a more compact form. We denote $F_{i-1} = 1 - g\left(\sum_{j=i}^{n} p_j\right)$ and $S_{i-1} = g\left(\sum_{j=i}^{n} p_j\right)$ for $i = 1, ..., n+1$, so $F_{i-1} = 1 - S_{i-1}$. Recall that $F_0 = 0$ and $S_n = 0$, then

$$\sum_{i=2}^{k} (x_{i-1} - x_i) F_{i-1} = \sum_{i=1}^{k-1} x_i (F_i - F_{i-1}) - x_k F_{k-1}$$

and,

$$\sum_{i=k+1}^{n} (x_i - x_{i-1}) S_{i-1} = \sum_{i=k+1}^{n} x_i (S_{i-1} - S_i) - x_k S_k.$$

The previous expressions applied to $\mathcal{C}_{g\circ\mathcal{P}}(X)$ lead to

$$\begin{aligned}\mathcal{C}_{g\circ\mathcal{P}}(X) = & \sum_{i=1}^{k-1} x_i\left(F_i - F_{i-1}\right) - x_k F_{k-1} + x_k + \sum_{i=k+1}^{n} x_i\left(S_{i-1} - S_i\right) - x_k S_k = \\ = & \sum_{i=1}^{n} x_i\left(S_{i-1} - S_i\right) = \sum_{i=1}^{n} x_i\left[g\left(\sum_{j=i}^{n} p_j\right) - g\left(\sum_{j=i+1}^{n} p_j\right)\right].\end{aligned} \tag{3.6}$$

If $g = id$, then $\rho_{id}(X) = \mathbb{E}(X)$. The same result for a continuous random variable is easy to prove using the definition of distortion risk measure and Fubinni's theorem. Expression (3.6) is useful to prove the following two propositions.

Proposition 3.2 (OWA equivalence to distortion risk measures)**.** *Let X be a discrete finite random variable and $\left(N, 2^N, \mathcal{P}\right)$ be a probability space as defined in proposition 3.1. Then there exist a unique $OWA_{\vec{w}_{\mathbf{x}}}$ operator such that $\rho_g(X) = OWA_{\vec{w}_{\mathbf{x}}}(\vec{x})$. The OWA operator is defined by weights*

$$w_{\mathbf{x},i} = g\left(\sum_{j=i}^{n} p_{\mathbf{x},j}\right) - g\left(\sum_{j=i+1}^{n} p_{\mathbf{x},j}\right), \quad i = 1, ..., n,$$

where $p_{\mathbf{x},j}$ is the probability of x_j for all j.

Note that this result shows that any distortion risk measure defines a set of OWA operators indexed by the discrete and finite random variables Γ defined on $\left(N, 2^N, \mathcal{P}\right)$,

$$\rho_g \leftrightarrow \mathcal{S}_{g,\Gamma} := \left\{OWA_{\vec{w}_{\mathbf{x}}} | X \in \Gamma\right\}.$$

Therefore, random variable X must be fixed to obtain a one-to-one equivalence between a distortion risk measure and an OWA operator.

Proposition 3.3 (WOWA equivalence to distortion risk measures)**.** *Let X be a discrete finite random variable and $\left(N, 2^N, \mathcal{P}\right)$ be a probability space as in proposition 3.1. If ρ_g is a distortion risk measure defined on this probability space, consider the $WOWA_{\vec{v},\vec{q}}$ operator such that $h = g$, $\vec{q} = \vec{p}_{\mathbf{x}}$ and $v_i = g\left(\frac{n-i+1}{n}\right) - g\left(\frac{n-i}{n}\right)$ for all $i = 1, ..., n$. Then*

$$\rho_g(X) = WOWA_{g,\vec{v},\vec{p_{\mathbf{x}}}}(\vec{x}),$$

where $\vec{p}_{\mathbf{x}}$ is the vector of probabilities of random variable X, and we explicitely indicate in the subindex that $WOWA_{\vec{v},\vec{p_{\mathbf{x}}}}$ depends on g

Proof. From proposition 3.2 it is known that there exists a unique $\vec{w}_{\mathbf{x}} \in [0,1]^n$ such that $OWA_{\vec{w}_{\mathbf{x}}}(\vec{x}) = \rho_g(X)$:

$$\begin{aligned}w_{\mathbf{x},i} = & \; g\left(\sum_{j=i}^{n} p_j\right) - g\left(\sum_{j=i+1}^{n} p_j\right) = \\ = & \; g\left(S_X\left(x_{i-1}\right)\right) - g\left(S_X\left(x_i\right)\right).\end{aligned} \tag{3.7}$$

In addition, there exists an $OWA_{\vec{u}_{\mathbf{x}}}$ operator such that $OWA_{\vec{u}_{\mathbf{x}}} = WOWA_{g,\vec{v},\vec{p}_{\mathbf{x}}}$ defined by

$$\begin{aligned} u_{\mathbf{x},i} = \; & g\left(\sum_{\Omega_j \in A_{id,i}} p_{\mathbf{x},j}\right) - g\left(\sum_{\Omega_j \in A_{id,i+1}} p_{\mathbf{x},j}\right) = \\ = \; & g\left(S_X\left(x_{i-1}\right)\right) - g\left(S_X\left(x_i\right)\right). \end{aligned} \tag{3.8}$$

Expressions (3.7) and (3.8) show that $\vec{w}_{\mathbf{x}} = \vec{u}_{\mathbf{x}}$ and, due to the uniqueness of the OWA operator, we conclude that $\rho_g(X) = OWA_{\vec{w}_{\mathbf{x}}}(\vec{x}) = WOWA_{g,\vec{v},\vec{p}_{\mathbf{x}}}(\vec{x})$, where $v_i = g\left(\frac{n-i+1}{n}\right) - g\left(\frac{n-i}{n}\right)$. □

Like in the case of the OWA operator, the relationship of the WOWA operator and distorsion risk measure ρ_g is such that the distortion risk measure can be interpreted as a set of WOWA operators indexed by the discrete and finite random variables defined on $\left(N, 2^N, \mathcal{P}\right)$, $\rho_g \leftrightarrow \mathcal{T}_{g,\Gamma} := \{WOWA_{g,\vec{v},\vec{p}_{\mathbf{x}}} | X \in \Gamma\}$. Again, the one-to-one equivalence between a distortion risk measure and a WOWA operator is obtained given that the random variable is fixed.

To summarize the results, for a given distortion function g and a discrete and finite random variable, there are three alternative ways to calculate the distortion risk measure that lead to the same result than using definition 2.5:

1. by means of the Choquet integral of X with respect to $\mu = g \circ \mathcal{P}$ using expression 3.6,
2. applying the $OWA_{\vec{w}_{\mathbf{x}}}$ operator to $\vec{x}$, following definition 2.7 with $w_{\mathbf{x},i} = g\left(\sum_{j=i}^{n} p_{\mathbf{x},j}\right) - g\left(\sum_{j=i+1}^{n} p_{\mathbf{x},j}\right)$, $\quad i = 1, ..., n$, and $p_{\mathbf{x},j}$ the probability of x_j for all j,
3. and, finally, applying the $WOWA_{g,\vec{v},\vec{p}_{\mathbf{x}}}$ operator to $\vec{x}$, following definition 2.9, where $v_i = g\left(\frac{n-i+1}{n}\right) - g\left(\frac{n-i}{n}\right)$ and $p_{\mathbf{x},j}$ the probability of x_j for all j.

3.1 Interpreting the concept of orness

Finally, we can derive another interesting application from expression (3.6). In particular, the concept of degree of orness introduced for the OWA operator may be formally extended to the case of $\mathcal{C}_{g \circ \mathcal{P}}(X)$, as:

$$orness\left(\mathcal{C}_{g \circ \mathcal{P}}(X)\right) := \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot \left[g\left(S_X\left(x_{i-1}\right)\right) - g\left(S_X\left(x_i\right)\right)\right]. \tag{3.9}$$

Note that this result is similar to (2.1). This result is now applicable to both positive and negative values and only the distorted probabilities are considered among capacities.

Let us show risk management applications of the degree of orness of the distortion risk measures. Note, for instance, that the regulatory requirements on risk measurement based on distortion risk measures may be reinterpreted by means of the degree of orness. Given a finite and discrete random variable X, when distortion risk measure $\rho_g(X)$ is required, the regulator has an implicit *preference weighting rule* with respect to the values of X that takes into account probabilities. This preference weighting rule can be summarized by $orness\,(OWA_{\vec{w}_{\mathbf{x}}})$, where $\vec{w}_{\mathbf{x}}$ is such that $w_{\mathbf{x},i} = g(S_X(x_{i-1})) - g(S_X(x_i))$. There are some cases of special interest, such as the mathematical expectation, the VaR_α and $TVaR_\alpha$ risk measures.

If $g = id$, then $\mathcal{C}_{g\circ\mathcal{P}} \equiv \mathbb{E}$ and

$$\begin{aligned} orness\,(\mathbb{E}(X)) := & \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot [S_X(x_{i-1}) - S_X(x_i)] = \\ = & \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot [p_{\mathbf{x},i}] = \sum_{i=1}^{n} \left(\frac{i}{n-1}\right) \cdot p_{\mathbf{x},i} - \frac{1}{n-1}. \end{aligned} \tag{3.10}$$

In particular, if the probability is discrete and uniform, i.e. $p_{\mathbf{x},i} = \frac{1}{n}$, then its orness is 1/2.

The degrees of orness of the VaR_α and $TVaR_\alpha$ distortion risk measures are obtained as follows. Given a confidence level $\alpha \in (0,1)$, let $k_\alpha \in \{1, 2, ..., n\}$ be such that $x_{k_\alpha} = \inf\{x_i | F_X(x_i) \geq \alpha\} = \sup\{x_i | S_X(x_i) \leq 1-\alpha\}$, i.e. x_{k_α} is the $\alpha-$quantile of X. From Table 2.1 $\psi_\alpha(S_X(x_i)) = \mathbb{1}_{(1-\alpha,1]}(S_X(x_i))$ for VaR_α and $\gamma_\alpha(S_X(x_i)) = \min\left\{\frac{S_X(x_i)}{1-\alpha}, 1\right\}$ for $TVaR_\alpha$, and then

$$\begin{aligned} VaR_\alpha(X): \quad & \psi_\alpha(S_X(x_{i-1})) - \psi_\alpha(S_X(x_i)) = \begin{cases} 0 & i < k_\alpha \\ 1 & i = k_\alpha \\ 0 & i > k_\alpha \end{cases} \\ TVaR_\alpha(X): \quad & \gamma_\alpha(S_X(x_{i-1})) - \gamma_\alpha(S_X(x_i)) = \begin{cases} 0 & i < k_\alpha \\ 1 - \dfrac{1}{1-\alpha} \displaystyle\sum_{j=k_\alpha+1}^{n} p_{\mathbf{x},j} & i = k_\alpha \\ \dfrac{p_{\mathbf{x},i}}{1-\alpha} & i > k_\alpha. \end{cases} \end{aligned} \tag{3.11}$$

From expressions (3.6) and (3.11) the degrees of orness of VaR_α and $TVaR_\alpha$ can easily be obtained as:

$$\begin{aligned} orness\,(VaR_\alpha(X)) = & \sum_{i=1}^{n} \left(\frac{i-1}{n-1}\right) \cdot [\psi_\alpha(S_X(x_{i-1})) - \psi_\alpha(S_X(x_i))] = \\ = & \frac{k_\alpha - 1}{n-1} \end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
orness\left(TVaR_{\alpha}\left(X\right)\right) = & \sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)\cdot\left[\gamma_{\alpha}\left(S_X\left(x_{i-1}\right)\right)-\gamma_{\alpha}\left(S_X\left(x_i\right)\right)\right] = \\
= & \left(\frac{k_{\alpha}-1}{n-1}\right)\cdot\left[1-\frac{1}{1-\alpha}\sum_{j=k_{\alpha}+1}^{n}p_{\mathbf{x},j}\right]+\sum_{i=k_{\alpha}+1}^{n}\left(\frac{i-1}{n-1}\right)\cdot\frac{p_{\mathbf{x},i}}{1-\alpha} = \\
= & \frac{k_{\alpha}-1}{n-1}+\frac{1}{1-\alpha}\cdot\sum_{i=k_{\alpha}+1}^{n}\left(\frac{i-k_{\alpha}}{n-1}\right)p_{\mathbf{x},i}.
\end{aligned}
\tag{3.13}
$$

So orness is directly connected to the α level chosen for the risk measure, i.e. the value of the distribution function at the point given by the value of the risk measure. In other words, for the Tail Value-at-Risk, orness indicates which new α^* level would be necessary for the Value at Risk to obtain the value of Tail Value-at-Risk at the initial α level.

The degree of orness of the distortion risk measure may be then calculated as a measure of the aggregation preference.

4 Illustrative example

A numerical example is provided to illustrate previous concepts. Let us consider two random variables X and Y such that $x_i = y_i$ for all i and $(x_1, x_2, x_3, x_4, x_5) = (-2, -1, 0, 1, 2)$. Table 4.1 summarizes the probabilities of both random variables.

Table 4.1: Summary of the probabilities of both rv

i	$x_i = y_i$	$p_{\mathbf{x},i}$	$F_X(x_i)$	$S_X(x_i)$	$p_{\mathbf{y},i}$	$F_Y(y_i)$	$S_Y(y_i)$
0	$-\infty$		0	1		0	1
1	-2	0.18	0.18	0.82	0.14	0.14	0.86
2	-1	0.31	0.49	0.51	0.11	0.25	0.75
3	0	0.225	0.715	0.285	0.72	0.97	0.03
4	1	0.25	0.965	0.035	0.02	0.99	0.01
5	2	0.035	1	0	0.01	1	0

We can calculate distortion risk measures for X and Y using aggregation operators. In particular, we are interested in $\mathbb{E}$, VaR_{α} and $TVaR_{\alpha}$ for certain $\alpha \in (0, 1)$. Using expression (3.6) and ψ_{α} and γ_{α} as in Table 2.1 we obtain expressions (4.1) and (4.2) for VaR_{α} and $TVaR_{\alpha}$.

$$VaR_\alpha(X) = \begin{cases} x_1 \cdot \left[1 - \mathbb{1}_{(1-\alpha,1]}(p_2+p_3+p_4+p_5)\right] + \\ x_2 \cdot \left[\mathbb{1}_{(1-\alpha,1]}(p_2+p_3+p_4+p_5) - \mathbb{1}_{(1-\alpha,1]}(p_3+p_4+p_5)\right] + \\ x_3 \cdot \left[\mathbb{1}_{(1-\alpha,1]}(p_3+p_4+p_5) - \mathbb{1}_{(1-\alpha,1]}(p_4+p_5)\right] + \\ x_4 \cdot \left[\mathbb{1}_{(1-\alpha,1]}(p_4+p_5) - \mathbb{1}_{(1-\alpha,1]}(p_5)\right] + \\ x_5 \cdot \left[\mathbb{1}_{(1-\alpha,1]}(p_5) - 0\right] \end{cases} \tag{4.1}$$

$$TVaR_\alpha(X) = \begin{cases} x_1 \cdot \left[1 - \min\left\{\frac{(p_2+p_3+p_4+p_5)}{1-\alpha}, 1\right\}\right] + \\ x_2 \cdot \left[\min\left\{\frac{(p_2+p_3+p_4+p_5)}{1-\alpha}, 1\right\} - \min\left\{\frac{(p_3+p_4+p_5)}{1-\alpha}, 1\right\}\right] + \\ x_3 \cdot \left[\min\left\{\frac{(p_3+p_4+p_5)}{1-\alpha}, 1\right\} - \min\left\{\frac{(p_4+p_5)}{1-\alpha}, 1\right\}\right] + \\ x_4 \cdot \left[\min\left\{\frac{(p_4+p_5)}{1-\alpha}, 1\right\} - \min\left\{\frac{(p_5)}{1-\alpha}, 1\right\}\right] + \\ x_5 \cdot \left[\min\left\{\frac{(p_5)}{1-\alpha}, 1\right\} - 0\right] \end{cases} \tag{4.2}$$

In this example we work with a confidence level $\alpha = 95\%$. In addition to these risk figures, the weighting vectors linked to the OWA and WOWA operators may be deduced for each distortion risk measure and random variable. Results are displayed in Tables 4.2 and 4.3.

Table 4.2: Summary of distortion risk measures and their associated distorted probabilities

	p_i		$\mathbb{E}$		$VaR_{95\%}$		$TVaR_{95\%}$	
$x_i = y_i$	X	Y	X	Y	X	Y	X	Y
-2	0.18	0.14	0.18	0.14	0	0	0	0
-1	0.31	0.11	0.31	0.11	0	0	0	0
0	0.225	0.72	0.225	0.72	0	1	0	0.4
1	0.25	0.02	0.25	0.02	1	0	0.3	0.4
2	0.035	0.01	0.035	0.01	0	0	0.7	0.2
Risk value			-0.35	-0.35	1	0	1.7	0.8
Degree of orness			0.4125	0.4125	0.75	0.5	0.925	0.7

Some comments may be made. First, observe that point probabilities ($p_i \mapsto w_i$) are distorted and a weighted average of the random values with respect to this distortion ($OWA_{\vec{w}}$) is calculated to obtain the distortion risk measures. Second, the degree of orness of a distortion risk measure can be understood as another risk measure for the random variable, with a value that belongs to $[0, 1]$. The information about riskiness that degree of

Table 4.3: Summary of WOWA vectors

rv	WOWA vectors	Distortion risk measure	Distortion function
X	$\vec{p}_{\mathbf{x}} = (18\%, 31\%, 22.5\%, 25\%, 3.5\%)$ $\vec{v} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	$\mathbb{E}$	id
	$\vec{p}_{\mathbf{x}} = (18\%, 31\%, 22.5\%, 25\%, 3.5\%)$ $\vec{v} = (0, 0, 0, 0, 1)$	$VaR_{95\%}$, $TVaR_{95\%}$	$\psi_{95\%}$, $\gamma_{95\%}$
Y	$\vec{p}_{\mathbf{y}} = (14\%, 11\%, 72\%, 2\%, 1\%)$ $\vec{v} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	$\mathbb{E}$	id
	$\vec{p}_{\mathbf{y}} = (14\%, 11\%, 72\%, 2\%, 1\%)$ $\vec{v} = (0, 0, 0, 0, 1)$	$VaR_{95\%}$, $TVaR_{95\%}$	$\psi_{95\%}$, $\gamma_{95\%}$

orness provides when these two random variables are compared is similar to the riskiness information deduced by their respective distortion risk measures. Third, the results show that weights $\vec{v}$ for the WOWA represent the risk attitude. In this example, we are only worried about the maximum loss when we consider $VaR_{95\%}$ and $TVaR_{95\%}$. By contrast, all values have the same importance in the case of the mathematical expectation. Note that weights take into account how the random variable is distributed by means of $\vec{p}_{\mathbf{x}}$ ($\vec{p}_{\mathbf{y}}$, respectively).

Observe that $VaR_{95\%}$ and $TVaR_{95\%}$ have equal $\vec{v}$ and $\vec{p}$ for each random variable, although the distortion risk measures take different values. It is due to the fact that h function in WOWA plays an important role to determine the particular distortion risk measure that is calculated. As shown in Table (2.1), $h = \psi_{95\%}$ in the case of $VaR_{95\%}$ and $h = \gamma_{95\%}$ in the case of $TVaR_{95\%}$ are two clearly different functions. Finally, we want to emphasize that the one-to-one relationship between a distortion risk measure and $OWA_{\vec{w}}$ operator depends on the random variable. It would be easy to check that the $OWA_{\vec{w}}$ linked to $VaR_{95\%}(X)$ matches with the one linked to $VaR_{99\%}(Y)$.

5 Discussion and conclusions

This article shows that distortion risk measures, OWA and WOWA operators in the discrete finite case are mathematically linked by means of the Choquet Integral. Aggregation operators are used as a natural form to introduce human subjectivity in decision making. From the risk management point of view, our main contribution is that we show how distortion risk measures may be derived -and then computed- from ordered weighted averaging operators. The mathematical links presented in this paper may help to interpret distortion risk measures under the fuzzy systems perspective. We show that the aggregation preference of the expert may be measured by means of the degree of orness of the distortion risk measure. Regulatory capital requirements and provisions may then be associated to the aggregation attitude of the regulator and the risk managers, respectively.

In our opinion, the mathematical link between risk concepts and fuzzy systems concepts presented in this paper offers a new line of research to be explored, that could provide a new perspective in quantitative risk management.

Appendix 1

Proof of OWA uniqueness
Given two different vectors $\vec{w}$ and $\vec{u}$ from $[0,1]^n$ we wonder if $OWA_{\vec{w}} = OWA_{\vec{u}}$, i,e, that the respective OWA operators on $\mathbb{R}^n$ are the same. We show that this is not possible. Suppose that, for all $\vec{x} \in \mathbb{R}^n$, $OWA_{\vec{w}}(\vec{x}) = OWA_{\vec{u}}(\vec{x})$. In particular, if we consider elements $\vec{z}_k \in \mathbb{R}^n, k = 1, ..., n$ defined by

$$\vec{z}_{k,i} = \begin{cases} 0 & if \quad i < k \\ 1/(n-i+1) & if \quad i \geq k \end{cases}$$

Then, going from $k = n$ to $k = 1$, we have that:

- **Step** $k = n$. We have $\vec{z}_n = (0, 0, ..., 0, 1)$, and permutation $\sigma = id$ is useful to calculate $OWA_{\vec{w}}(\vec{z}_n)$ and $OWA_{\vec{u}}(\vec{z}_n)$. Precisely, $OWA_{\vec{w}}(\vec{z}_n) = 1 \cdot w_n$ and $OWA_{\vec{u}}(\vec{z}_n) = 1 \cdot u_n$. If $OWA_{\vec{w}} = OWA_{\vec{u}}$, then $u_n = w_n$.
- **Step** $k = n-1$. We have $\vec{z}_{n-1} = \left(0, 0, ..., \frac{1}{2}, 1\right)$, and permutation $\sigma = id$ is still useful. So $OWA_{\vec{w}}(\vec{z}_{n-1}) = \frac{1}{2} \cdot w_{n-1} + 1 \cdot w_n$ and, taking into account the previous step, $OWA_{\vec{u}}(\vec{z}_{n-1}) = \frac{1}{2} \cdot u_{n-1} + 1 \cdot w_n$. If the hypothesis $OWA_{\vec{w}} = OWA_{\vec{u}}$ holds, then $u_{n-1} = w_{n-1}$.
- **Step** $k = i$. From previous steps we have that $u_j = w_j$, $j = i+1, ..., n$ and in this step we obtain $u_i = w_i$.
- **Step** $k = 1$. Finally, supposing again that $OWA_{\vec{w}} = OWA_{\vec{u}}$, we obtain that $u_j = w_j$ for all $j = 1, ..., n$. But this is a contradiction with the fact that $\vec{w} \neq \vec{u}$.

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